THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 8

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1. (Convergence and Continuity)

   (a) Define pointwise and uniform convergence of sequence of functions.

   (b) State the uniform convergence theorem for continuous functions.

   (c) State a criterion for proving that a sequence of functions does NOT converge uniformly.

   (d) We consider the following explicit examples:

      i. Show that $f_n(x) := \frac{x}{x+n}$ converges pointwisely to 0, for $x \in [0, \infty)$. The convergence is uniform on $[0, a]$ for any $a > 0$, but not uniform on $[0, \infty)$.

      ii. Evaluate the limit $f_n(x) := \frac{x^n}{1+x^n}$ for $x \geq 0$. Show that the convergence is uniform on $[0, b]$ for any $0 < b < 1$, but not uniform on $[0, 1]$.

      iii. Show that $f_n(x) : e^{-nx}$ converges to 0 uniformly on $[a, \infty)$ for any $a > 0$, but not uniform on $[0, \infty)$.

      iv. Show that $f_n(x) := xe^{-nx}$ converges to 0 uniformly on $[0, \infty)$.

   For each of the above examples, check whether the limit function is continuous and find the discontinuity points (if any).

   Proof for (iv). Equivalently we are going to show that

   $$\lim_{n\to\infty} \sup_{x\in[0,\infty)} f_n(x) = 0.$$

   Now $f_n'(x) = e^{-nx} + xe^{-nx}(-n)$, and by calculus we know that for each n, $f_n(x)$ attains its maximum at $x = \frac{1}{n}$, whence $f_n(x) = \frac{1}{n}e^{-1}$. Hence

   $$\lim_{n\to\infty} \sup_{x\in[0,\infty)} f_n(x) = \lim_{n\to\infty} \frac{1}{n}e^{-1} = 0.$$

   □

   Remark: The proof I used on Monday is more "heuristic" but not so rigourous. The above proof is easier to follow.

   (e) Let $f_n(x) := x + \frac{1}{n}$ and $f(x) := x$, $x \in [0, \infty)$. Then $f_n \to f$ uniformly on $[0, \infty)$. However, show that f_n^2 does not converge to x^2 uniformly on $[0, \infty)$. This shows that product of uniformly converging functions may not converge uniformly to the product of limits. The problem is that the functions f_n are unbounded.

(f) Let D be any domain and let $f_n(x), g_n(x) : D \to \mathbb{R}$ be uniformly bounded, that is, there is $M > 0$ so that for any $x \in D, n \in \mathbb{N}$, $|f_n(x)| \leq M$. Suppose f_n, g_n converges uniformly to f, g, respectively. Show that $f_n g_n$ converges to fg uniformly.

2. (Convergence and Riemann Integral)

   (a) Show that a sequence of Riemann integrable functions may converge pointwisely to a function which is not Riemann integrable.

   (b) State the uniform convergence theorem for Riemann integrable functions.

   (c) Examine whether $\int_a^b f_n \to \int_a^b f$ in the above examples, where $0 \leq a < b < \infty$. Hence find an example in which $f_n : [a, b] \to \mathbb{R}$ does NOT converge to f uniformly, yet $\int_a^b f_n \to \int_a^b f$.

3. (Reference, not discussed in tutorials) We consider the following examples for functions defined on $\mathbb{R}$: they all converge pointwisely to 0, but their Riemann integrals do not converge to 0. It should be understood using improper Riemann integrals.

   (a) (Concentration) Let f_n "converge to a delta-function", say, let $f_n(x)$ be defined by:

$$f_n(x) := \begin{cases} n^2 x + n, \text{ if } -\frac{1}{n} \leq x < 0 \\ -n^2 x + n, \text{ if } 0 < x \leq \frac{1}{n} \\ 0, \text{ otherwise} \end{cases}$$

   Then $f_n \to 0$ but $\int_{-1}^1 f_n = 1$ for each n.

   (b) (Diffusion) The process is reverse to the above. Let f_n "die down", say, let $f_n(x)$ be defined by:

$$f_n(x) := \begin{cases} n^{-2} x + \frac{1}{n}, \text{ if } -n \leq x < 0 \\ -n^{-2} x + \frac{1}{n}, \text{ if } 0 << x < n \\ 0,, \text{ otherwise} \end{cases}$$

   Then $f_n \to 0$ but $\int_{-\infty}^{\infty} f_n = 1$ for each n.

   (c) (Translation) Let $f : [0, 1]$ be a fixed function which is Riemann integrable, with $\int_0^1 f = 1$. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) := f(x - n).$$

   Then $f_n \to 0$ but $\int_{-\infty}^{\infty} f_n = 1$ for each n.